

# Partitioning the Boolean lattice into copies of a poset

Vytautas Gruslys\*

Imre Leader\*

István Tomon\*

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## Abstract

Let  $P$  be a poset of size  $2^k$  that has a greatest and a least element. We prove that, for sufficiently large  $n$ , the Boolean lattice  $2^{[n]}$  can be partitioned into copies of  $P$ . This resolves a conjecture of Lonc.

## 1 Introduction

Let  $2^{[n]}$  denote the *Boolean lattice* of dimension  $n$ , that is, the poset (partially ordered set) whose elements are the subsets of  $[n] = \{1, \dots, n\}$ , ordered by inclusion.

An important property of the Boolean lattice is that any finite poset  $P$  can be embedded into  $2^{[n]}$  for sufficiently large  $n$ . Here by an *embedding* of a poset  $P$  into a poset  $Q$  we mean an injection  $f : P \rightarrow Q$  such that  $f(x) \leq_Q f(y)$  if and only if  $x \leq_P y$ . For any embedding  $f : P \rightarrow Q$ , we call the image  $f(P)$  a *copy* of  $P$  in  $Q$ .

Now, if  $P$  is fixed and  $n$  is large, then  $2^{[n]}$  contains many copies of  $P$ . So a natural question arises: can  $2^{[n]}$  be partitioned into copies of  $P$ ? Of course, for such a partition to exist, the size of  $P$  must divide the size of  $2^{[n]}$ , that is,  $|P|$  must be a power of 2 (we would like to emphasise that we denote by  $|P|$  the number of elements of  $P$  and not the number of relations). Moreover,  $P$  must have a greatest and a least element. Lonc [8] conjectured that these obvious necessary conditions are in fact sufficient.

**Conjecture 1** (Lonc). *Let  $P$  be a poset of size  $2^k$  with a greatest and a least element. Then, for sufficiently large  $n$ , the Boolean lattice  $2^{[n]}$  can be partitioned into copies of  $P$ .*

The case where  $P$  is a chain of size  $2^k$  was originally conjectured by Sands [9]. Griggs [3] proposed a slightly stronger conjecture that, for any positive integer  $c$  and for sufficiently large  $n$ , it is possible to partition  $2^{[n]}$  into chains of length  $c$  and at most one other chain. Both conjectures were proved by Lonc [8]. The question of minimising the dimension  $n$  in Griggs' conjecture in terms of the length of the chain  $c$  has received attention from several authors, including Elzobi and Lonc [1] and Griggs, Yeh and Grinstead [4]. Recently, Tomon [11] proved that the smallest sufficient  $n$  is of order  $\Theta(c^2)$ . Related questions on partitioning  $2^{[n]}$  into chains of almost equal lengths have also been examined, by Füredi [2], Hsu, Logan, Shahriari and Towse [6, 7] and Tomon [10].

As we mentioned in the previous paragraph, Lonc himself verified Conjecture 1 in the case where  $P$  is a chain. Furthermore, it is easy to extend this result to products of chains.

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\*Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Wilberforce Road, CB3 0WA Cambridge, United Kingdom; e-mail: `{v.gruslys,i.leader,i.tomon}@cam.ac.uk`.

In fact, for any two posets  $P, Q$ , if  $2^{[n]}$  can be partitioned into copies of  $P$  and  $2^{[m]}$  can be partitioned into copies of  $Q$ , then  $2^{[n+m]}$  can be partitioned into copies of  $P \times Q$ . However, apart from some small cases that can be checked by hand, chains and their products were the only two cases for which Lonc's conjecture had been confirmed.

In this paper we resolve the conjecture in full generality.

**Theorem 2.** *Let  $P$  be a poset of size  $2^k$  with a greatest and a least element. Then, for sufficiently large  $n$ , the Boolean lattice  $2^{[n]}$  can be partitioned into copies of  $P$ .*

The plan of the paper is as follows. In Section 2 we give the most important definitions and outline the structure of the proof of Theorem 2. We give the actual proof in Sections 3 and 4: Section 3 contains a general argument, which works in various settings where a partition of a product set into smaller sets is sought, and might be of independent interest; Section 4 contains ideas that are particular to partitioning  $2^{[n]}$  into copies of a fixed poset. Finally, in Section 5 we give some open problems.

## 2 Overview of the proof

### 2.1 Weak partitions

A key idea in the proof will be the interplay between partitions and two weaker notions, called  $r$ -partitions and  $(1 \bmod r)$ -partitions, which we now describe. This idea appears (in a different context) in a paper by Gruslys, Leader and Tan [5].

Let  $P$  be a poset. Recall that a set  $A \subset 2^{[n]}$  is a *copy* of  $P$  if the poset induced on  $A$  by  $2^{[n]}$  is isomorphic to  $P$ . We define  $\mathcal{F}_n(P)$  to be the family of all copies of  $P$  in  $2^{[n]}$ .

Let  $X$  be a set, and let  $\mathcal{F}$  be any family of subsets of  $X$ . A *weight function* on  $\mathcal{F}$  is an assignment of non-negative integer weights to the members of  $\mathcal{F}$ . For an element  $x \in X$ , the *multiplicity* of  $x$  for a weight function is the total weight of those members of  $\mathcal{F}$  that contain  $x$ . So, for example,  $X$  can be partitioned into members of  $\mathcal{F}$  if and only if there exists a weight function on  $\mathcal{F}$  for which every element of  $X$  has multiplicity 1. For a positive integer  $r$ , we say that

- $\mathcal{F}$  contains an  $r$ -partition of  $X$  if there is a weight function on  $\mathcal{F}$  for which every element of  $X$  has multiplicity  $r$ ;
- $\mathcal{F}$  contains a  $(1 \bmod r)$ -partition of  $X$  if there is a weight function on  $\mathcal{F}$  for which every  $x \in X$  has multiplicity  $1 + rk_x$ , where  $k_x \in \{0, 1, \dots\}$  may depend on  $x$ .

Our strategy revolves around establishing a close relation between  $r$ -partitions,  $(1 \bmod r)$ -partitions and actual partitions of sets. Obviously, if  $\mathcal{F}$  contains a partition of  $X$ , then  $\mathcal{F}$  contains an  $r$ -partition and a  $(1 \bmod r)$ -partition of  $X$  for every  $r$ . Our aim is to go in the opposite direction. Namely, our strategy consists of two steps: firstly, we will show that if there exists an  $r$  such that  $\mathcal{F}$  contains an  $r$ -partition and a  $(1 \bmod r)$ -partition of  $X$ , then we can use these weak partitions to get an actual partition of  $X^m$  for some  $m$ ; secondly, we will show that, for some  $n$  and  $r$ ,  $\mathcal{F}_n(P)$  does contain an  $r$ -partition and a  $(1 \bmod r)$ -partition of  $2^{[n]}$ .

It is not immediately obvious that this strategy should work. For instance, it is not clear that finding weak partitions of  $2^{[n]}$  is easier than finding an actual partition. However, this will turn out to be the case in Section 4, where we prove the following lemmas.

**Lemma 3.** *Let  $P$  be a finite poset with a greatest and a least element. Then there exist positive integers  $n$  and  $r$  such that the family of copies of  $P$  in  $2^{[n]}$  contains an  $r$ -partition of  $2^{[n]}$ .*

**Lemma 4.** *Let  $P$  be a finite poset of size  $2^k$  that has a greatest and a least element, and let  $r$  be a positive integer. Then there exists a positive integer  $n$  such that the family of copies of  $P$  in  $2^{[n]}$  contains a  $(1 \bmod r)$ -partition of  $2^{[n]}$ .*

A key part of the argument will be to see how to use these seemingly much weaker results can be used to find an actual partition of  $2^{[n]}$ . We will discuss this in the following subsection.

## 2.2 Product systems

We will prove a very general theorem, which, applied to Lemmas 3 and 4, will imply our main result.

Let  $S$  be a set. For two sets  $A \subset S^m, B \subset S^n$  with  $m \leq n$ , we say that  $B$  is a *copy* of  $A$  if  $B$  can be obtained by taking a product of  $A$  with a singleton set in  $S^{n-m}$  and permuting the coordinates. More precisely, for a permutation  $\pi$  of  $\{1, \dots, n\}$  and  $x = (x_1, \dots, x_n) \in S^n$ , we define  $\pi(x) = (x_{\pi(1)}, \dots, x_{\pi(n)})$ . Moreover, for any  $X \subset S^n$ , we define  $\pi(X) = \{\pi(x) : x \in X\}$ . Finally, for any  $X \subset S^m$  and  $Y \subset S^{n-m}$ , we define  $X \times Y = \{(x_1, \dots, x_m, y_1, \dots, y_{n-m}) : (x_1, \dots, x_m) \in X, (y_1, \dots, y_{n-m}) \in Y\}$ . Note that we abuse the notation slightly and identify  $S^m \times S^{n-m}$  with  $S^n$ , which allows us to consider  $X \times Y$  as a subset of  $S^n$ . With these definitions,  $B$  is a copy of  $A$  if  $B = \pi(A \times \{y\})$  for some permutation  $\pi$  of  $\{1, \dots, n\}$  and some  $y \in S^{n-m}$ .

Note that this definition does not exactly agree with the definition of a copy of a poset, which we made in Section 1. Indeed, there may exist two sets  $A, B \subset 2^{[n]}$  such that  $2^{[n]}$  induces the same poset on  $A$  and  $B$ , but such that  $B$  cannot be obtained from  $A$  by permuting the coordinates. However, we think that this abuse of notation is not harmful, because it will always be clear from the context which definition of a copy should be used. Moreover, if sets  $A \subset 2^{[n]}$  and  $B \subset 2^{[m]}$  are copies in the new sense, then they are also copies when considered as posets. Therefore, the two definitions are in fact closely related.

The following theorem is vital for our strategy.

**Theorem 5.** *Let  $S$  be a finite set and let  $\mathcal{F}$  be a family of subsets of  $S$ . Suppose that there exists a positive integer  $r$  such that  $\mathcal{F}$  contains an  $r$ -partition and a  $(1 \bmod r)$ -partition of  $S$ . Then there exists a positive integer  $n$  such that  $S^n$  can be partitioned into copies of members of  $\mathcal{F}$ .*

This theorem was inspired by work of Gruslys, Leader and Tan [5]. They implicitly used a special case of this theorem to prove that, for any finite (non-empty) set  $T \subset \mathbb{Z}^k$ , there exists a positive integer  $n$  such that  $\mathbb{Z}^n$  can be partitioned into isometric copies of  $T$ .

It is straightforward to deduce our main theorem from Lemmas 3 and 4 and Theorem 5. Indeed, let  $P$  be a poset of size  $2^k$  with a greatest and a least element. Lemma 3 implies that there are positive integers  $r$  and  $u$  such that  $\mathcal{F}_u(P)$  contains an  $r$ -partition of  $2^{[u]}$ . Now Lemma 4 implies that there is a positive integer  $v$  such that  $\mathcal{F}_v(P)$  contains a  $(1 \bmod r)$ -partition of  $2^{[v]}$ . Setting  $m = \max\{u, v\}$ ,  $\mathcal{F}_m(P)$  contains both an  $r$ -partition and a  $(1 \bmod r)$ -partition of  $2^{[m]}$ . We can now apply Theorem 5 with  $\mathcal{F} = \mathcal{F}_m(P)$  and

$S = 2^{[m]}$  to finish the proof. (Note that if  $B \subset 2^{[mn]}$  is a copy of some  $A \in \mathcal{F}_m(P)$ , then the poset that  $2^{[mn]}$  induces on  $B$  is isomorphic to  $P$ , and hence  $B \in \mathcal{F}_{mn}(P)$ .)

### 3 Partitions in product systems

Our aim in this section is to prove Theorem 5.

**Theorem 5.** *Let  $S$  be a finite set and let  $\mathcal{F}$  be a family of subsets of  $S$ . Suppose that there exists a positive integer  $r$  such that  $\mathcal{F}$  contains an  $r$ -partition and a  $(1 \bmod r)$ -partition of  $S$ . Then there exists a positive integer  $n$  such that  $S^n$  can be partitioned into copies of members of  $\mathcal{F}$ .*

As in the statement of the theorem, we let  $\mathcal{F}$  be a family of subsets of a finite set  $S$  and we suppose that  $r$  is a natural number such that  $\mathcal{F}$  contains an  $r$ -partition and  $(1 \bmod r)$ -partition of  $S$ . The set  $S$ , family  $\mathcal{F}$  and number  $r$  will remain fixed throughout this section.

**Lemma 6.** *For any sets  $A, B \subset S$ , there exists a positive integer  $n$  such that  $S^2 \times (A \cup B)^n$  can be partitioned into copies of members of  $\mathcal{F} \cup \{A, B\}$ .*

The proof of Lemma 6 is by far the most complicated part of this paper. We will prove Lemma 6 in the next subsection. Now, with Lemma 6 at our disposal, we will prove Theorem 5.

**Proposition 7.** *Let  $A, B \subset S$  and suppose that there exist positive integers  $p, q$  such that*

- $S^p$  can be partitioned into copies of members of  $\mathcal{F} \cup \{A\}$ , and
- $S^2 \times A^q$  can be partitioned into copies of members of  $\mathcal{F} \cup \{B\}$ .

*Then  $S^{pq+2}$  can be partitioned into copies of members of  $\mathcal{F} \cup \{B\}$ .*

*Proof.* Partition  $S^p$  into sets  $X_1, \dots, X_u, Y_1, \dots, Y_v$ , where every  $X_i$  is a copy of  $A$  and every  $Y_j$  is a copy of a member of  $\mathcal{F}$ . We denote  $\mathcal{X} = \{X_1, \dots, X_u\}$  and  $\mathcal{Y} = \{Y_1, \dots, Y_v\}$ . Then  $S^{pq+2} = S^2 \times (S^p)^q$  is the disjoint union of sets  $S^2 \times Z_1 \times \dots \times Z_q$  with  $Z_i \in \mathcal{X} \cup \mathcal{Y}$  for all  $i$ . We separate these sets into two families, namely,

$$\begin{aligned} \mathcal{A} &= \{S^2 \times Z_1 \times \dots \times Z_q : Z_i \in \mathcal{X} \text{ for all } i\}, \\ \mathcal{B} &= \{S^2 \times Z_1 \times \dots \times Z_q : Z_i \in \mathcal{X} \cup \mathcal{Y} \text{ for all } i \text{ and } Z_j \in \mathcal{Y} \text{ for some } j\}. \end{aligned}$$

Each member of  $\mathcal{A}$  is a copy of  $S^2 \times A^q$ , so it can be partitioned into copies of members of  $\mathcal{F} \cup \{B\}$ . Moreover, each member of  $\mathcal{B}$  can be partitioned into copies of some member of  $\mathcal{F}$  in an obvious way. Since together these sets form a partition of  $S^{pq+2}$ , we are done.  $\square$

*Proof of Theorem 5 (assuming Lemma 6).* Since  $\mathcal{F}$  contains an  $r$ -partition of  $S$  with  $r \geq 1$ , and since  $S$  is finite, we can find finitely many sets  $B_1, \dots, B_k \in \mathcal{F}$  that cover  $S$ . We define  $A_i = B_1 \cup \dots \cup B_i$  for every  $1 \leq i \leq k$ . So, in particular,  $A_k = S$ .

We will use reverse induction on  $i$  to prove that there exist positive integers  $p_1, \dots, p_k$  such that, for every  $1 \leq i \leq k$ ,  $S^{p_i}$  can be partitioned into copies of members of  $\mathcal{F} \cup \{A_i\}$ . If  $i = k$ , then  $A_k = S$ , and the statement is trivially true with, say,  $p_k = 1$ . So we may

assume that  $1 \leq i \leq k-1$ . Since  $A_{i+1} = A_i \cup B_{i+1}$ , it follows from Lemma 6 that there exists a positive integer  $q$  such that  $S \times (A_{i+1})^q$  can be partitioned into copies of members of  $\mathcal{F} \cup \{A_i, B_{i+1}\}$ . However,  $B_{i+1}$  is a member of  $\mathcal{F}$ , so  $\mathcal{F} \cup \{A_i, B_{i+1}\} = \mathcal{F} \cup \{A_i\}$ . Combining this with the induction hypothesis for  $i+1$  and Proposition 7, we see that  $S^{p_i}$ , where  $p_i = p_{i+1}q + 2$ , can be partitioned into copies of members of  $\mathcal{F} \cup \{A_i\}$ .

In particular, the statement holds for  $i = 1$ . Since  $A_1 = B_1 \in \mathcal{F}$ , it says that  $S^{p_1}$  can be partitioned into copies of members of  $\mathcal{F}$ , as required.  $\square$

### 3.1 Proof of Lemma 6

Here we will prove Lemma 6.

**Lemma 6.** *For any sets  $A, B \subset S$ , there exists a positive integer  $n$  such that  $S^2 \times (A \cup B)^n$  can be partitioned into copies of members of  $\mathcal{F} \cup \{A, B\}$ .*

We start by picking two sets  $A, B \subset S$ ; these sets will be fixed throughout the subsection. We define  $U = A \cup B$ ,  $\bar{A} = U \setminus A$  and  $\bar{B} = U \setminus B$ . Moreover, for any integers  $1 \leq i \leq d$ , we define

$$C_{i,d} = \underbrace{\bar{A} \times \cdots \times \bar{A} \times \overset{\substack{i\text{-th component} \\ \downarrow}}{\bar{B}} \times \bar{A} \times \cdots \times \bar{A}}_{d \text{ components}}.$$

We also define  $C_{0,d} = \bar{A}^d$ . Our aim is to prove that there exists a positive integer  $n$  such that  $S \times U^n$  can be partitioned into copies of members of  $\mathcal{F} \cup \{A, B\}$ .

At certain points in the proof we will be conjuring up extra elbow space by ‘blowing up’  $S^k$ , for some  $k$ , into  $S^{k+1}$ . It turns out that sometimes a set  $X \subset S^k$  can be usefully identified with a larger set  $X \times \bar{A} \subset S^{k+1}$ . The following simple proposition is an example of this idea.

**Proposition 8.** *Let  $k \geq 1$  and let  $X \subset U^k$  be such that  $U^k \setminus X$  can be partitioned into copies of  $A$  and  $B$ . Then  $U^{k+1} \setminus (X \times \bar{A})$  can be partitioned into copies of  $A$  and  $B$ .*

*Proof.* Partition  $U^{k+1} \setminus (X \times \bar{A})$  into sets  $(U^k \setminus X) \times \bar{A}$  and  $U^k \times A$ ; the first of these sets can be partitioned into copies of  $U^k \setminus X$ , and the second – into copies of  $A$ .  $\square$

If we could prove that  $U^k$ , for some  $k$ , can be partitioned into copies of  $A$  and  $B$  (that is, without using  $\mathcal{F}$ ), then we would be done. Of course, this is not possible in general. However, we can partition  $U^k$  with one  $C_{i,k}$  removed.

**Proposition 9.** *For any integers  $k \geq 1$  and  $0 \leq i \leq k$ , the set  $U^k \setminus C_{i,k}$  can be partitioned into copies of  $A$  and  $B$ .*

*Proof.* We use induction on  $k$ . If  $k = 1$ , then, depending on the value of  $i$ ,  $U \setminus C_{i,1}$  is either  $A$  or  $B$ . If  $k \geq 2$ , we may assume that  $i \neq k$  (in fact, there are only two distinct cases:  $i = 0$  and  $i \neq 0$ ). By the induction hypothesis,  $U^{k-1} \setminus C_{i,k-1}$  can be partitioned into copies of  $A$  and  $B$ . However,  $C_{i,k} = C_{i,k-1} \times \bar{A}$ , so we are done by Proposition 8.  $\square$

Proposition 8 says that if we can partition a subset of  $U^k$ , then we can also partition an ‘equivalent’ subset of  $U^{k+1}$ . The following proposition allows us to use the extra space in  $U^{k+1}$  to slightly modify this subset.

**Proposition 10.** Let  $X \subset U^k$  be such that  $U^k \setminus X$  can be partitioned into copies of  $A$  and  $B$ . Suppose that  $X$  contains the set  $C_{i,k}$  for some  $0 \leq i \leq k$ . Then the set  $U^{k+1} \setminus Y$ , where

$$Y = (X \times \bar{A}) \cup C_{k+1,k+1} \setminus C_{i,k+1},$$

can also be partitioned into copies of  $A$  and  $B$ .

*Proof.* Partition  $U^{k+1} \setminus Y$  into four sets  $Z_1, Z_2, Z_3, Z_4$ , where

$$\begin{aligned} Z_1 &= (U^k \setminus C_{0,k}) \times \bar{B}, \\ Z_2 &= (U^k \setminus C_{i,k}) \times (A \cap B), \\ Z_3 &= C_{i,k} \times B, \\ Z_4 &= (U^k \setminus X) \times \bar{A}. \end{aligned}$$

It is evident from Figure 1 that these four sets do partition  $U^{k+1} \setminus Y$ . The sets  $Z_1$  and

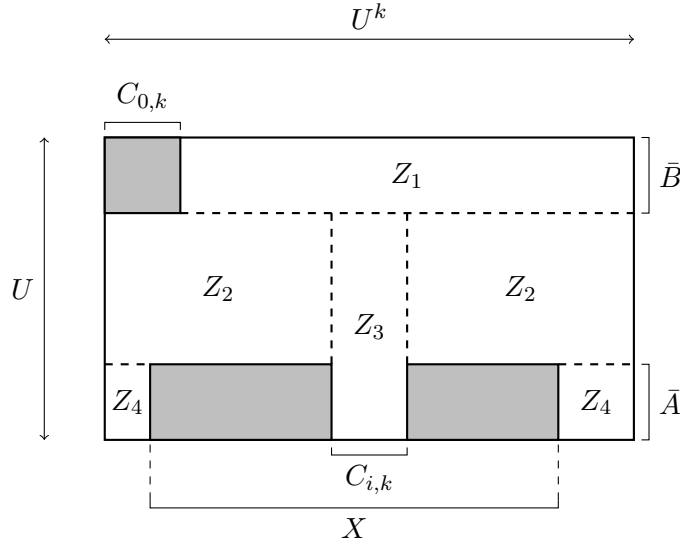


Figure 1: The set  $Y \subset U^{k+1}$  is shaded. The four sets  $Z_1, Z_2, Z_3, Z_4$  partition  $U^{k+1} \setminus Y$ .

$Z_2$  can be partitioned into copies of  $A$  and  $B$  by Proposition 9. The set  $Z_3$  is obviously a union of disjoint copies of  $B$ . Finally,  $Z_4$  is a union of disjoint copies of  $U^k \setminus X$ , so it can be partitioned into copies of  $A$  and  $B$  by the assumption on  $X$ .  $\square$

The previous proposition enables us to make one change to the set  $X$  when we go one dimension up, that is, from  $U^k$  to  $U^{k+1}$ . To make multiple changes, we apply this proposition multiple times. This is exactly the content of Corollary 11.

**Corollary 11.** Let  $k, l$  be non-negative integers and let  $I \subset \{0, \dots, k\}, J \subset \{k+1, \dots, k+l\}$  be sets such that  $|J| = |I|$ . Then the set  $U^{k+l} \setminus Y$ , where

$$Y = (U^k \times \bar{A}^l) \cup \left( \bigcup_{j \in J} C_{j,k+l} \right) \setminus \left( \bigcup_{i \in I} C_{i,k+l} \right),$$

can be partitioned into copies of  $A$  and  $B$ .

*Proof.* We shall apply induction on  $l$ . If  $l = 0$ , then  $|J| = |I| = 0$ , so  $U^k \setminus Y = \emptyset$ , and hence the conclusion trivially holds.

Now suppose that  $l \geq 1$ . We will split the argument into two cases, depending on whether or not  $k + l \in J$ . If  $k + l \in J$ , then we write  $j^* = k + l$  and we pick any  $i^* \in I$ . We define  $I^* = I \setminus \{i^*\}$  and  $J^* = J \setminus \{j^*\}$ . Finally, we define

$$Y^* = \left( U^k \times \bar{A}^{l-1} \right) \cup \left( \bigcup_{j \in J^*} C_{j, k+l-1} \right) \setminus \left( \bigcup_{i \in I^*} C_{i, k+l-1} \right).$$

By the induction hypothesis,  $U^{k+l-1}$  can be partitioned into copies of  $A$  and  $B$ . Moreover,  $Y = (Y^* \times \bar{A}) \cup C_{k+l, k+l} \setminus C_{i^*, k+l}$ , so we can apply Proposition 10 to finish the proof in this case.

On the other hand, if  $k + l \notin J$ , then we define

$$Y' = \left( U^k \times \bar{A}^{l-1} \right) \cup \left( \bigcup_{j \in J} C_{j, k+l-1} \right) \setminus \left( \bigcup_{i \in I} C_{i, k+l-1} \right)$$

and observe that  $Y = Y' \times \bar{A}$ . Moreover,  $U^{k+l-1} \setminus Y'$  can be partitioned into copies of  $A$  and  $B$  by the induction hypothesis, and hence it follows from Proposition 8 that the same holds for  $U^{k+l} \setminus Y$ .  $\square$

Recall that our ultimate goal in this subsection is to partition  $S^2 \times U^n$ , for some  $n \geq 1$ , into copies of members of  $\mathcal{F} \cup \{A, B\}$ . We cannot achieve this goal just yet, but we have already provided ourselves with tools, in the form of Propositions 8 to 10 and Corollary 11, that allow us to partition  $U^k \setminus X$ , for various  $k$  and various sets  $X$ , into copies of  $A$  and  $B$ . Our strategy now can be roughly described as follows. We will take a large  $n$  and we will slice  $S^2 \times U^n$  up into copies of  $S \times U^n$ . We will partition big parts of these slices into copies of members of  $\mathcal{F} \cup \{A, B\}$ , leaving out gaps that we can control. Then we will combine the gaps across all slices, and we will fill them in with copies of members of  $\mathcal{F}$ . The following proposition will tell us what gaps we should leave in the slices so that their union could be filled in later on.

**Proposition 12.** *Let  $t$  be a positive integer and take not necessarily distinct sets  $P_1, \dots, P_t \in \mathcal{F}$ . Define  $Q_0, \dots, Q_t \subset S \times U^t$  by setting*

$$Q_i = \begin{cases} P_i \times C_{i,t} & \text{if } 1 \leq i \leq t, \\ S \times C_{0,t} & \text{if } i = 0. \end{cases}$$

*Then the set  $(S \times U^t) \setminus (Q_0 \cup \dots \cup Q_t)$  can be partitioned into copies of members of  $\mathcal{F} \cup \{A \cup B\}$ .*

*Proof.* We use induction on  $t$ . We take  $t = 0$  to be the base case. Although the set  $C_{0,0}$  had not been defined, we may interpret  $S \times C_{0,0}$  and  $S \times U^0$  as both being the set  $S$ , in which case the conclusion says that the empty set can be partitioned into copies of members of  $\mathcal{F} \cup \{A, B\}$ , which is trivially true.

Now suppose that  $t \geq 1$ . We write  $X = Q_0 \cup \dots \cup Q_t$  and  $X^* = (S \times C_{0,t-1}) \cup (P_1 \times C_{1,t-1}) \cup \dots \cup (P_{t-1} \times C_{t-1,t-1})$ . By the induction hypothesis,  $(S \times U^{t-1}) \setminus X^*$  can be

partitioned into copies of  $\mathcal{F} \cup \{A, B\}$ . Moreover, using the fact that  $X = (X^* \times \bar{A}) \cup Q_t$ , we can partition  $(S \times U^t) \setminus X$  into three sets  $Y_1, Y_2, Y_3$ , where

$$\begin{aligned} Y_1 &= ((S \times U^{t-1}) \setminus X^*) \times \bar{A}, \\ Y_2 &= ((S \times U^{t-1}) \setminus (P_t \times C_{0,t-1})) \times A, \\ Y_3 &= P_t \times U^{t-1} \times (A \cap B). \end{aligned}$$

It is clear from Figure 2 that these sets do partition  $(S \times U^t) \setminus X$ . Moreover,  $Y_1$  can be partitioned into copies of members of  $\mathcal{F} \cup \{A, B\}$  (by the induction hypothesis);  $Y_2$  is trivially a disjoint union of copies of  $A$ ;  $Y_3$  is a disjoint union of copies of  $P_t$ , which is a member of  $\mathcal{F}$ .  $\square$

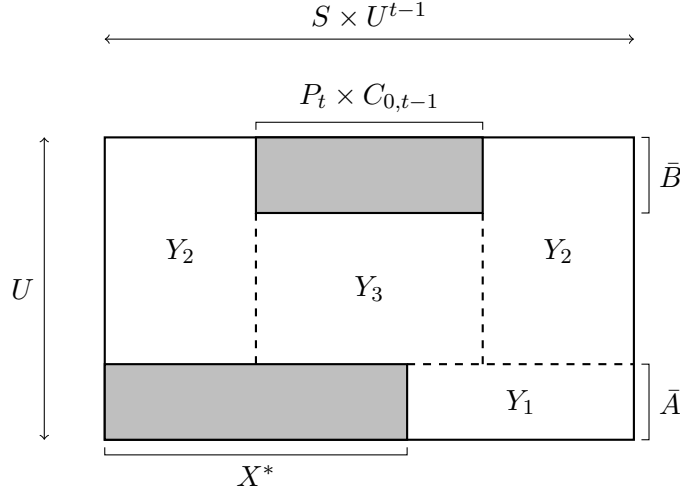


Figure 2: The set  $X$  is shaded;  $Y_1, Y_2, Y_3$  partition  $(S \times U^{t-1}) \setminus X$ .

We recall that, for some positive integer  $r$ ,  $\mathcal{F}$  contains an  $r$ -partition of  $S$ . In other words, there exist not necessarily distinct sets  $P_1, \dots, P_m \in \mathcal{F}$  such that every element of  $S$  is contained in precisely  $r$  of them. We will use the sets  $P_1, \dots, P_m$  to prove the following proposition.

**Proposition 13.** *Let  $r$  be as above. For any positive integer  $k$  there exists an integer  $l \geq k$  with the following property. For any distinct numbers  $j_1, \dots, j_t \in \{1, \dots, l\}$ , if  $t \leq k$  and  $t \equiv 1 \pmod{r}$ , then the set  $S \times (U^l \setminus \bigcup_{u=1}^t C_{j_u, l})$  can be partitioned into copies of members of  $\mathcal{F} \cup \{A, B\}$ .*

*Proof (see Figure 3).* Given  $k$ , fix any  $l \geq k + (k-1)m/r$ . Given distinct  $j_1, \dots, j_t \in \{1, \dots, l\}$ , we may assume (after a permutation of coordinates, if necessary), that  $\{j_1, \dots, j_t\} = \{l-t+1, \dots, l\}$ . We denote this set by  $J$ . Since  $t \leq k$  and  $t \equiv 1 \pmod{r}$  by assumption, we may write  $t = ar + 1$  for some integer  $0 \leq a \leq (k-1)/r$ . We will prove that the set

$$Y = S \times \left( U^l \setminus \bigcup_{j \in J} C_{j, l} \right)$$



can be partitioned into copies of members of  $\mathcal{F} \cup \{A, B\}$ .

Extend  $P_1, \dots, P_m$  to a longer list  $P_1, \dots, P_{am}$  by setting  $P_{i+m} = P_i$  for every  $m+1 \leq i \leq am$ . The only important property of this new list is that every member of the original list is repeated exactly  $a$  times. Moreover, set  $P_0 = S$ . Then every element of  $S$  is contained in exactly  $ar + 1 = t$  members of the list  $P_0, \dots, P_{am}$ . We define

$$X = (S \times U^{am} \times \bar{A}^{l-am}) \setminus \left( \bigcup_{i=0}^{am} P_i \times C_{i,l} \right).$$

Since  $X = ((S \times U^{am}) \setminus (\bigcup_{i=0}^{am} P_i \times C_{i,am})) \times \bar{A}^{l-am}$ , it follows from Proposition 12 that  $X$  can be partitioned into copies of members of  $\mathcal{F} \cup \{A, B\}$ . Since  $\min J > l - k \geq am$ , the set  $X$  is disjoint from  $S \times C_{j,l}$  for any  $j \in I$ , and hence  $X \subset Y$ . Therefore, it only remains to prove that  $Y \setminus X$  can be partitioned into copies of members of  $\mathcal{F} \cup \{A, B\}$ .

For any  $z \in S$ , we denote by  $S_z$  the cross-section of  $Y \setminus X$  at  $z$ , that is,

$$Y_z = \{y \in U^l : (z, y) \in Y \setminus X\}.$$

For the moment, let us focus on one fixed  $z \in S$ . By construction of  $P_0, \dots, P_{am}$ , there are exactly  $t$  values of  $i$  for which  $z \in P_i$ . Let  $I$  be the set of these values. Then

$$Y_z = U^l \setminus \left( \left( U^{am} \times \bar{A}^{l-am} \right) \cup \left( \bigcup_{j \in J} C_{j,l} \right) \setminus \left( \bigcup_{i \in I} C_{i,l} \right) \right).$$

Since  $|I| = |J| = t$ ,  $I \subset \{0, \dots, am\}$  and  $J \subset \{am+1, \dots, l\}$ , Corollary 11 implies that  $Y_z$  can be partitioned into copies of members of  $\mathcal{F} \cup \{A, B\}$ .

Now we are done:  $Y = X \cup (\bigcup_{z \in S} \{z\} \times Y_z)$ , and we have proved that  $X$  and every  $Y_z$  can be partitioned into copies of members of  $\mathcal{F} \cup \{A, B\}$ .  $\square$

We are now ready to prove Lemma 6.

*Proof of Lemma 6.* We begin by recalling that  $\mathcal{F}$  contains a  $(1 \bmod r)$ -partition of  $S$ . In other words, there exists a family of not necessarily distinct sets  $R_1, \dots, R_k \in \mathcal{F}$  such that every  $x \in S$  is contained in exactly  $1 + ra_x$  members of this family, where  $a_x$  is an integer. Furthermore, Proposition 13 provides us with a positive integer  $n \geq k$  such that, for any set  $I \subset \{1, \dots, n\}$  that satisfies  $|I| \equiv 1 \pmod{r}$  and  $|I| \leq k$ , the set  $S \times (U^n \setminus \bigcup_{i \in I} C_{i,n})$  can be partitioned into copies of members of  $\mathcal{F} \cup \{A, B\}$ . We will show that  $S \times U^n$  can be partitioned into copies of members of  $\mathcal{F} \cup \{A, B\}$ .

We define

$$X = (S \times U^n) \setminus \left( \bigcup_{i=1}^k R_i \times C_{i,n} \right)$$

and, for any  $y \in S$ , we let  $X_y$  denote the cross-section of  $X$  at  $y$ , that is,  $X_y = \{x \in U^n : (y, x) \in X\}$ . Any  $y \in S$  is contained in  $1 + ra_y$  members of the family  $R_1, \dots, R_k$ . Therefore, if we write  $J_y = \{j \in [k] : y \in R_j\}$ , then  $|J_y| \equiv 1 \pmod{r}$  and  $|J_y| \leq k$ . Moreover, it is easy to see that

$$X_y = U^n \setminus \left( \bigcup_{j \in J_y} C_{j,n} \right).$$

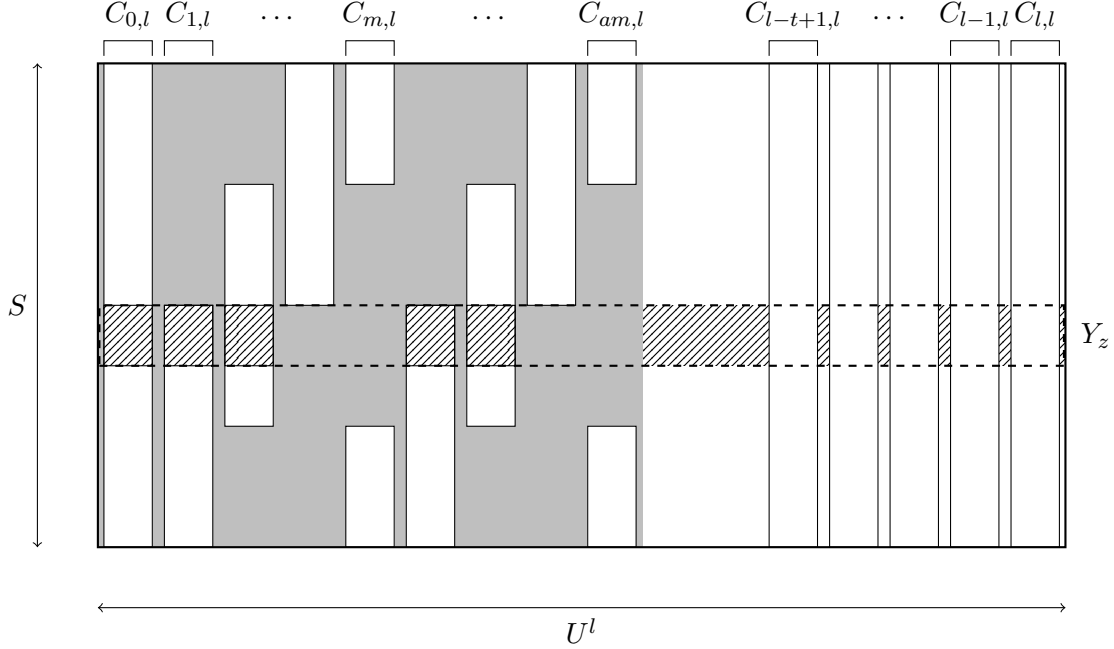


Figure 3: The set  $X$  is shaded, a slice  $Y_z$  is hatched diagonally. Proposition 12 and Corollary 11, respectively, imply that  $X$  and  $Y_z$  can be partitioned into copies of members of  $\mathcal{F} \cup \{A, B\}$ .

By Proposition 13,  $S \times X_y$  can be partitioned into copies of members of  $\mathcal{F} \cup \{A, B\}$ . Therefore, so can be  $S \times X$ , which is the disjoint union of sets  $S \times \{y\} \times X_y$ ,  $y \in S$ .

Finally, observe that  $S^2 \times U^n$  is the disjoint union of  $S \times X$  and sets  $S \times R_i \times C_{i,n}$ ,  $1 \leq i \leq k$ . Each set  $S \times R_i \times C_{i,n}$  is trivially a union of disjoint copies of  $R_i$ , which is a member of  $\mathcal{F}$ . Therefore,  $S^2 \times U^n$  can be partitioned into copies of members of  $\mathcal{F} \cup \{A, B\}$ , as required.  $\square$

## 4 Weak partitions

### 4.1 Constructing an $r$ -partition of $2^{[n]}$

Our aim in this subsection is to prove Lemma 3, which asserts the existence of an  $r$ -partition of  $2^{[n]}$  into copies of  $P$  for some  $n, r$ . Our proof is somewhat technical, but not very difficult.

Recall that by our earlier definition a weight function is an assignment of non-negative integer weights to sets from some selected family. We now extend this definition to allow more general weights. Namely, given a set  $V \subset \mathbb{R}$  and a set family  $\mathcal{F}$ , a  $V$ -valued weight function on  $\mathcal{F}$  is a function  $w : \mathcal{F} \rightarrow V$ . Usually, we will take  $V$  to be  $\mathbb{Z}$ ,  $\mathbb{Z}^+$  or  $\mathbb{Q}^+$ , where  $S^+$  is defined to be  $S \cap [0, \infty)$  for any  $S \subset \mathbb{R}$ . We note that a weight function in the old sense is precisely a  $\mathbb{Z}^+$ -valued weight function in the new sense.

Moreover, if  $\mathcal{F}$  is a family of subsets of some set  $X$ , for any  $x \in X$  we define the multiplicity of  $x$  for  $w$ , denoted  $N_w(x)$ , to be the total weight assigned to the members of

$\mathcal{F}$  that contain  $x$ . That is,

$$N_w(x) = \sum_{\substack{A \in \mathcal{F} \\ x \in A}} w(A).$$

Moreover, for any  $Y \subset X$ , we set  $N_w(Y) = \sum_{y \in Y} N_w(y)$ . With these definition at hand, we can restate Lemma 3 in a form that is slightly more convenient for the proof.

**Lemma 3'.** *Let  $P$  be a finite poset with a greatest and a least element. Then there exist a positive integer  $n$  and a  $\mathbb{Q}^+$ -valued weight function  $w$  on the copies of  $P$  in  $2^{[n]}$  such that  $N_w(x) = 1$  for all  $x \in 2^{[n]}$ .*

To see why Lemma 3' is equivalent to Lemma 3, observe that a  $\mathbb{Q}^+$ -valued weight function  $w$  on a finite set family  $\mathcal{F}$  can be made into a  $\mathbb{Z}^+$ -valued weight function by multiplying it by the least common multiple of the denominators of the  $w(A)$  for  $A \in \mathcal{F}$ . Moreover, if  $N_w(x) = 1$  for all  $x$ , then the resulting  $\mathbb{Z}^+$ -valued weight function  $rw$  satisfies  $N_{rw}(x) = r$  for all  $x$ .

The main idea in the proof is to look for a weight function that is symmetric with respect to all permutations of the ground set  $\{1, \dots, n\}$ . Such a weight function can be obtained by averaging any another weight function over all permutations of  $\{1, \dots, n\}$ . This idea essentially removes the need to consider the structure of the poset  $P$ , and converts Lemma 3' into a question about finding a certain weight function on the power set of  $\{0, \dots, n\}$ . This is reflected in the following definition.

Let  $P$  be a poset and  $n$  a positive integer. Moreover, let  $w$  be a  $\mathbb{Q}^+$ -valued weight function on the copies of  $P$  in  $2^{[n]}$ . We define a new  $\mathbb{Q}^+$ -valued weight function  $w^{\text{sym}}$ , also on the copies of  $P$  in  $2^{[n]}$ , by setting

$$w^{\text{sym}}(A) = \frac{1}{n!} \sum_{\pi \in \text{Perm}(n)} w(\pi(A))$$

for all  $A$  that are copies of  $P$  in  $2^{[n]}$ . Here  $\text{Perm}(n)$  denotes the set of permutations of  $\{1, \dots, n\}$  and we recall that  $\pi(A)$  denotes the image of  $A$  after permuting the coordinates of  $2^{[n]}$  according to  $\pi$ .

Since elements of  $2^{[n]}$  are subsets of  $\{1, \dots, n\}$ , it makes sense to write  $|x|$  for  $x \in 2^{[n]}$  to denote the size of  $x$ . We partition  $2^{[n]}$  into levels  $L_0, \dots, L_n$ , where  $L_k = \{x \in 2^{[n]} : |x| = k\}$ . Then, for any  $x \in L_k$ ,

$$N_{w^{\text{sym}}}(x) = \frac{1}{\binom{n}{k}} N_w(L_k).$$

Therefore, our task is reduced to finding  $w$  such that  $N_w(L_k) = \binom{n}{k}$  for all  $k$ . To this aim, we would like to have a tool for embedding  $P$  into  $2^{[n]}$  while keeping control on levels into which we map the elements of  $P$ . The following proposition provides us with such a tool.

We say that a set  $A \subset \mathbb{Z}$  is *d-scattered* if, for any distinct  $i, j \in A$ , we have  $|i - j| \geq d$ .

**Proposition 14.** *Let  $P$  be a finite poset with a greatest and a least element. Then there exists a positive integer  $d$  such that, for any integer  $n \geq (|P| - 1)d$  and any  $d$ -scattered set  $A \subset \{0, \dots, n\}$  of size  $|P|$ , there exists an embedding  $\phi : P \rightarrow 2^{[n]}$  satisfying*

$$\{|\phi(x)| : x \in P\} = A.$$

In other words, for any  $0 \leq k \leq n$ ,

$$|L_k \cap \phi(P)| = \begin{cases} 1 & \text{if } k \in A, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We start by recalling that, since  $P$  is finite, it can be embedded into  $2^{[k]}$  for some  $k$ . Let  $\psi : P \rightarrow 2^{[k]}$  be an embedding which maps the greatest element of  $P$  to the greatest element of  $2^{[k]}$  and the least element of  $P$  to the least element of  $2^{[k]}$ . We write  $s = |P|$  and list the elements of  $P$  as  $p_1, \dots, p_s$  in the order where  $0 = |\psi(p_1)| \leq \dots \leq |\psi(p_s)| = k$ .

We will prove that  $d = k$  works. Indeed, take any integer  $n \geq (s-1)k$  and let  $A \subset \{0, \dots, n\}$  be a  $k$ -scattered set of size  $s$ . Then  $A = \{a_1, \dots, a_s\}$ , where  $0 \leq a_1 < \dots < a_s \leq n$  and  $a_{i+1} \geq a_i + k$  for all  $0 \leq i \leq s-1$ . For every  $1 \leq i \leq s$ , we set

$$\phi(p_i) = \psi(p_i) \cup \{k+1, \dots, k+a_i - |\psi(p_i)|\}.$$

To prove that  $\phi : P \rightarrow 2^{[n]}$  is a well-defined embedding, we have to check that  $0 \leq a_1 - |\psi(p_1)| \leq \dots \leq a_s - |\psi(p_s)| \leq n - k$ . However, if we prove this, then it is trivial to see that  $|\phi(p_i)| = a_i$  for all  $i$ , as required.

First, we observe that  $a_1 - |\psi(p_1)| = a_1 \geq 0$  and  $a_s - |\psi(p_s)| = a_s - k \leq n - k$ . Furthermore, for any  $1 \leq i \leq s-1$ , we have  $a_{i+1} - |\psi(p_{i+1})| \geq a_i + k - k = a_i \geq a_i - |\psi(p_i)|$ , and so we are done.  $\square$

**Proposition 15.** *Let  $X$  be a finite set and  $t$  a positive integer. If  $f : X \rightarrow \mathbb{Q}^+$  is a function such that*

$$t \max_{x \in X} f(x) \leq \sum_{x \in X} f(x),$$

*then there exists a  $\mathbb{Q}^+$ -valued weight function  $w$  on the family of  $t$ -element subsets of  $X$ , such that  $N_w(x) = f(x)$  for all  $x \in X$ .*

*Proof.* Let  $r$  be the least common multiple of the denominators of the  $f(x)$  over all  $x \in X$ . After multiplying  $f$  by  $tr$ , we may assume that  $f$  takes values in  $\mathbb{Z}^+$  and that  $\sum_{x \in X} f(x)$  is divisible by  $t$ . We denote  $\sum_{x \in X} f(x) = Nt$  and we will use induction on  $N$ .

If  $f(x) = 0$  for all  $x \in X$ , then the result is trivial. Therefore, we may assume that  $N \geq 1$ . Let  $S = \{x \in X : f(x) > 0\}$  and  $T = \{x \in X : f(x) = N\}$ . Since

$$t \max_{x \in X} f(x) \leq \sum_{x \in X} f(x) \leq |S| \max_{x \in X} f(x),$$

it follows that  $|S| \geq t$ . Moreover,  $N|T| \leq \sum_{x \in X} f(x) = Nt$ , and hence  $|T| \leq t$ . Therefore, there exists a set  $A$  such that  $T \subset A \subset S$  and  $|A| = t$ .

We define  $g : X \rightarrow \mathbb{Z}^+$  by setting

$$g(x) = \begin{cases} f(x) - 1 & \text{if } x \in A, \\ f(x) & \text{otherwise.} \end{cases}.$$

Then  $\sum_{x \in X} g(x) = (N-1)t$  is non-negative and divisible by  $t$ . Moreover, since  $T \subset A$ , we have  $g(x) \leq N-1$  for all  $x \in X$ . Therefore, by the induction hypothesis, there exists a

$\mathbb{Q}^+$ -valued weight function  $w'$  on the  $t$ -element subsets of  $X$ , such that  $N_{w'}(x) = g(x)$  for all  $x \in X$ . We define

$$w(B) = \begin{cases} w'(A) + 1 & \text{if } B = A, \\ w'(B) & \text{if } B \subset X, |B| = t \text{ and } B \neq A. \end{cases}$$

This  $w$  satisfies the required conditions.  $\square$

It is easy to deduce Lemma 3' from Propositions 14 and 15.

*Proof of Lemma 3'.* Let  $P$  be a finite poset with a greatest and a least element. Recall that our aim is to find, for some positive integer  $n$ , a  $\mathbb{Q}^+$ -valued weight function  $w$  on the copies of  $P$  in  $2^{[n]}$ , such that  $N_w(L_i) = \binom{n}{i}$  for all  $0 \leq i \leq n$ . Indeed, then  $N_{w^{\text{sym}}}(x) = 1$  for all  $x \in 2^{[n]}$ .

Let  $d$  be such that, for any  $n \geq (|P| - 1)d$  and any  $d$ -scattered set  $A \subset \{0, \dots, n\}$  of size  $|P|$ , there exists a copy of  $P$  in  $2^{[n]}$ , say  $C$ , such that  $\{|x| : x \in C\} = A$ . The existence of such a number  $d$  is guaranteed by Proposition 14. Set  $k = |P|d$ .

Choose  $n$  large enough to satisfy the inequality  $k \binom{n}{\lceil n/2 \rceil} \leq 2^n$ . Then Proposition 15 gives a  $\mathbb{Q}^+$ -valued weight function  $w'$  on the  $k$ -element subsets of  $\{0, \dots, n\}$  that satisfies  $N_{w'}(i) = \binom{n}{i}$  for all  $0 \leq i \leq n$ .

Let  $B$  be a  $k$ -element subset of  $\{0, \dots, n\}$ . If we consider the elements of  $B$  in increasing order and take every  $d$ th element, we obtain a  $d$ -scattered set. In this way we can partition  $B$  into  $d$ -scattered sets  $B_1, \dots, B_d$ , each of size  $k/d = |P|$ . We say that  $B$  *splits* into sets  $B_1, \dots, B_d$ .

By splitting  $k$ -element sets we obtain a  $\mathbb{Q}^+$ -valued weight function  $w''$  on  $d$ -scattered  $|P|$ -element subsets of  $\{0, \dots, n\}$ . More precisely, we define  $w''(A) = \sum w'(B)$ , summing over all  $k$ -element sets  $B \subset \{0, \dots, n\}$  with the property that  $A$  is one of the sets into which  $B$  splits. Note that we have  $N_{w''}(i) = N_{w'}(i) = \binom{n}{i}$  for all  $0 \leq i \leq n$ .

Finally, for any  $d$ -scattered  $|P|$ -element set  $A \subset \{0, \dots, n\}$  we choose one copy of  $P$  in  $2^{[n]}$ , denoted  $C_A$ , such that  $\{|x| : x \in C_A\} = A$ . We define a  $\mathbb{Q}^+$ -valued weight function  $w$  on the copies of  $P$  in  $2^{[n]}$  by setting

$$w(C) = \begin{cases} w''(A) & \text{if } C = C_A \text{ for some } d\text{-scattered } |P|\text{-element set } A \subset \{0, \dots, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

We note that every  $d$ -scattered  $|P|$ -element set  $A \subset \{0, \dots, n\}$  contributes  $w''(A)$  towards both  $N_{w''}(i)$  and  $N_w(L_i)$  for every  $i \in A$ , and 0 towards both  $N_{w''}(j)$  and  $N_w(L_j)$  for every  $j \notin A$ . Therefore,  $N_w(L_i) = N_{w''}(i) = \binom{n}{i}$  for all  $0 \leq i \leq n$ , as required.  $\square$

## 4.2 Constructing a $(1 \bmod r)$ -partition of $2^{[n]}$

Here we prove Lemma 4, which asserts the existence of an  $(1 \bmod r)$ -partition of  $2^{[n]}$  into copies of  $P$  for some  $n$ . This proof is shorter, but slightly trickier than that of Lemma 3. We begin by recasting Lemma 4 in a form which is stronger, but more convenient to work with.

**Lemma 4'.** *Let  $P$  be a poset of size  $2^k$  with a greatest and a least element. Then there exist a positive integer  $n$  and a  $\mathbb{Z}$ -valued weight function  $w$  on the copies of  $P$  in  $2^{[n]}$  satisfying  $N_w(x) = 1$  for all  $x \in 2^{[n]}$ .*

We remark that Lemma 4' does imply Lemma 4, because the  $\mathbb{Z}$ -valued weight function  $w$  can be converted into a suitable  $\mathbb{Z}^+$ -valued weight function  $w'$  by choosing  $w'(A) \in \{0, \dots, r-1\}$  such that  $w'(A) \equiv w(A) \pmod{r}$ , for all  $A$ .

*Proof of Lemma 4'.* Since  $P$  is finite, it can be embedded into  $2^{[d]}$ , for some  $d$ , by an embedding which maps the greatest and the least elements of  $P$  to the corresponding elements of  $2^{[d]}$ . We will show that  $n = 2d - 1$  works.

We say that a function  $f : 2^{[n]} \rightarrow \mathbb{Z}$  is *realisable* if there exists a  $\mathbb{Z}$ -valued weight function  $w$  on the copies of  $P$  in  $2^{[n]}$ , such that  $N_w(x) = f(x)$  for all  $x \in 2^{[n]}$ . We note that if  $f, g$  are realisable functions, then so are  $f + g$  and  $f - g$ . Our aim is to show that the constant 1 function on  $2^{[n]}$  is realisable.

For any  $A \subset 2^{[n]}$ , we define  $1_A : 2^{[n]} \rightarrow \{0, 1\}$  to be the indicator function of  $A$ . Clearly, if  $A$  is a copy of  $P$ , then  $1_A$  is realisable.

We denote the greatest and the least elements of  $2^{[n]}$  by  $x_+, x_-$ . Let  $x \in 2^{[n]}$ . If  $|x| \geq d$ , then there exists an embedding  $2^{[d]} \rightarrow 2^{[n]}$  which maps the greatest element of  $2^{[d]}$  to  $x$ . Therefore, in  $2^{[n]}$ , we can find a copy of  $P$  whose greatest element is  $x$ . We denote this copy by  $A$ . Moreover, if we denote  $B = A \setminus \{x\}$ , then  $B \cup \{x\}$  and  $B \cup \{x_+\}$  are copies of  $P$ . Therefore, the function  $1_{\{x\}} - 1_{\{x_+\}} = 1_{B \cup \{x\}} - 1_{B \cup \{x_+\}}$  is realisable.

Similarly, if  $|x| \leq d$ , then there exists an embedding  $2^{[d]} \rightarrow 2^{[n]}$  which maps the least element of  $2^{[d]}$  to  $x$ . Then we can find a copy of  $P$  in  $2^{[n]}$ , which we denote by  $A$ , with the property that  $x$  is the least element of  $A$ . We write  $B = A \setminus \{x\}$  and observe that  $A \cup \{x\}$  and  $A \cup \{x_-\}$  are copies of  $P$ . Therefore, the function  $1_{\{x\}} - 1_{\{x_-\}} = 1_{B \cup \{x\}} - 1_{B \cup \{x_-\}}$  is realisable.

In particular, for any  $x \in 2^{[n]}$ , at least one of the functions  $1_{\{x\}} - 1_{\{x_+\}}$  and  $1_{\{x\}} - 1_{\{x_-\}}$  is realisable. Moreover, if  $|x| = d$ , then both of them are. Therefore, by choosing any  $x_0 \in 2^{[n]}$  with  $|x_0| = d$ , we can see that  $1_{\{x_+\}} - 1_{\{x_-\}} = (1_{\{x_0\}} - 1_{\{x_-\}}) - (1_{\{x_0\}} - 1_{\{x_+\}})$  is realisable. We conclude that, in fact, for any  $x, y \in 2^{[n]}$ , the function  $1_{\{x\}} - 1_{\{y\}}$  is realisable.

Let  $f, g : 2^{[n]} \rightarrow \mathbb{Z}$  be two functions that satisfy  $\sum_{x \in 2^{[n]}} f(x) = \sum_{x \in 2^{[n]}} g(x)$ . Then the difference  $f - g$  can be expressed as a sum of functions of the form  $1_{\{x\}} - 1_{\{y\}}$  with  $x, y \in 2^{[n]}$ , so  $f - g$  is realisable. Hence,  $f$  is realisable if and only if  $g$  is realisable. Therefore, to prove that the constant 1 function is realisable, it is enough to find one realisable function  $f$  such that  $\sum_{x \in 2^{[n]}} f(x) = 2^n$ . However, we know that  $|P| = 2^k$  and, trivially,  $k \leq n$ , so we can take  $f = 2^{n-k} \cdot 1_A$  for any  $A \subset 2^{[n]}$  which is a copy of  $P$ .  $\square$

## 5 Concluding remarks and open problems

In the proof of Theorem 2 we do not explicitly keep track of a value of  $n$  that would be sufficient. This is to make the proof more readable. Moreover, we did not put any serious effort into finding a good bound. The following bound can be extracted from the proof.

**Theorem 2'.** *There exists an absolute constant  $C > 0$  with the following property. Let  $P$  be a poset of size  $2^k$  with a greatest and a least element. Then, for any integer  $n \geq 2^{\lfloor P \rfloor^C}$ , the Boolean lattice  $2^{[n]}$  can be partitioned into copies of  $P$ .*

It is interesting to ask what happens if  $P$  does not satisfy the conditions required by Theorem 2. Of course, then it is impossible to partition  $2^{[n]}$  into copies of  $P$ . However, what if we are allowed to leave a small number of elements of  $2^{[n]}$  uncovered? For example,

if  $P$  does not have a greatest and/or a least element, then the greatest and/or the least element of  $2^{[n]}$  are the only ones that obviously cannot be covered by copies of  $P$ . Lonc [8] conjectured that, if  $n$  is large and if an obvious divisibility condition is satisfied, then  $2^{[n]}$  with its greatest and least element removed can be partitioned into copies of  $P$ .

**Conjecture 16** (Lonc). *Let  $P$  be a finite poset. If  $n$  is sufficiently large and if  $|P|$  divides  $2^n - 2$ , then it is possible to partition  $2^{[n]}$ , with its greatest and least element removed, into copies of  $P$ .*

In the spirit of Griggs' conjecture it is reasonable to hope that, even if we do not impose any divisibility conditions for  $|P|$ , for sufficiently large  $n$ ,  $2^{[n]}$  can be partitioned into copies of  $P$  and a set of size  $c$ , where  $c < |P|$ . Or perhaps one can bound  $c$  by a weaker constant which depends on  $P$ .

**Question 17.** *Let  $P$  be a finite poset. Must there exist a constant  $c = c(P)$  such that, for any  $n$ , it is possible to cover all but at most  $c$  elements of  $2^{[n]}$  by disjoint copies of  $P$ ?*

We remark that Conjecture 16 would give a positive answer to Question 17 in the case where  $|P|$  is not a multiple of 4.

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